Estimates for periodic Zakharov-Shabat operators

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Abstract

We consider the periodic Zakharov-Shabat operators on the real line. The spectrum of this operator consists of intervals separated by gaps with the lengths $|g_n|\geqslant 0, n\in\mathbb{Z}$. Let $\mu_n^\pm$ be the corresponding effective masses and let h_n be heights of the corresponding slits in the quasimomentum domain. We obtain a priori estimates of sequences $g=(|g_n|)_{n\in\mathbb{Z}}, \mu^\pm=(\mu_n^\pm)_{n\in\mathbb{Z}}, h=(h_n)_{n\in\mathbb{Z}}$ in terms of weighted ℓ^p-norms at $p\geqslant 1$. The proof is based on the analysis of the quasimomentum as the conformal mapping.

1 Introduction and main results

Consider the Zakharov-Shabat operator $\mathcal{T}$ acting on $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ and given by

$$\mathcal{T}=\mathcal{J}\frac{d}{dt}+V(t),\qquad V=\begin{pmatrix} V_1 & V_2\\ V_2 & -V_1\end{pmatrix}\ ,\ \ \mathcal{J}=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},$$

where V is a real 1-periodic 2×2 matrix-valued function of $t\in\mathbb{R}$ and $V\in L^1(0,1)$. In order to describe our main result we shall introduce some notations and recall some well known facts about the Zakharov-Shabat operator (see [Kr], [LS] for details). The spectrum of $\mathcal{T}$ is purely absolutely continuous and is given by the set $\cup\sigma_n$, with spectral bands $\sigma_n=[z_{n-1}^+,z_n^-]$, where $\ldots<z_{2n-1}^-\leqslant z_{2n-1}^+<z_{2n}^-\leqslant z_{2n}^+<\ldots$ and $z_n^\pm=n(\pi+o(1))$ as $|n|\to\infty$. These intervals σ_n,σ_{n+1} are separated by a gap $g_n=(z_n^-,z_n^+)$ with length $|g_n|\geqslant 0$. If a gap g_n is degenerate, i.e. $g_n=\emptyset$, then the corresponding spectral bands σ_n,σ_{n+1} merge. The sequence $\ldots<z_{2n-1}^-\leqslant z_{2n-1}^+<z_{2n}^-\leqslant z_{2n}^+<\ldots$ is the spectrum of equation $\mathcal{J}f'+Vf=zf$ with the 2-periodic boundary conditions, i.e., $f(t+2)=f(t), t\in\mathbb{R}$. Here the equality $z_n^-=z_n^+$ means that z_n^- is the double eigenvalue. The eigenfunctions, corresponding to the eigenvalue $z_n^\pm$, are 1-periodic, when n is even and they are antiperiodic, i.e., $f(t+1)=-f(t),\ t\in\mathbb{R}$, when n is odd. Introduce the 2×2-matrix valued fundamental solution $\psi=\psi(t,z)$ of the problem:

$$\mathcal{J}\psi'+V\psi=z\psi,\quad \psi(0,z)=\mathcal{I}_2,\quad z\in\mathbb{C}, \tag{1.1}$$

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where $\mathcal{I}_2$ is the identity 2×2-matrix. Introduce the Lyapunov function $\Delta(z) = \frac{1}{2}\mathrm{Tr}\psi(1,z)$. Note that $\Delta(z_n^{\pm}) = (-1)^n, n \in \mathbb{Z}$, and the function $\Delta'(z)$ has exactly one zero $z_n \in [z_n^-, z_n^+]$ for each $n \in \mathbb{Z}$. For each V there exists a unique conformal mapping (the quasi-momentum) $k : \mathcal{Z} \to K(h)$ such that (see [Mi1])

$$\cos k(z) = \Delta(z), \quad z \in \mathcal{Z} = \mathbb{C} \setminus \cup \bar{\gamma}_n, \qquad K(h) = \mathbb{C} \setminus \cup \Gamma_n, \quad \Gamma_n = (\pi n - ih_n, \pi n + ih_n),$$

$$k(z) = z + o(1) \qquad as \quad |z| \to \infty,$$

where Γ_n is the vertical slit and the height $h_n \geqslant 0$ is defined by the equation $\cosh h_n = (-1)^n \Delta(z_n) \geqslant 1$. We emphasize that the introduction of the quasi-momentum $k(z)$ provides a natural labeling of all gaps γ_n (including the empty ones!) by demanding that $k(\cdot)$ maps the gap g_0 on the vertical slit $\Gamma_0 = (-ih_0, ih_0)$.

For any $p \geqslant 1$ and the weight $\omega = (\omega_n)_{n\in\mathbb{Z}}$, where $\omega_n \geqslant 1$, we introduce the real spaces

$$\ell^p_\omega = \{f = (f_n)_{n\in\mathbb{Z}} : \ \ \|f\|_{p,\omega} < \infty\}, \qquad \|f\|^p_{p,\omega} = \sum_{n\in\mathbb{Z}} \omega_n f_n^p < \infty.$$

If the weight $\omega_n = 1$ for all $n \in \mathbb{Z}$, then we will write $\ell^p_0 = \ell^p$ with the norm $\| \cdot \|_p$. For each V we introduce the sequences

$$h = (h_n)_{n\in\mathbb{Z}}, \quad \gamma = (|\gamma_n|)_{n\in\mathbb{Z}}, \quad J = (J_n)_{n\in\mathbb{Z}}, \quad J_n = |A_n|^{\frac{1}{2}} \geqslant 0, \quad A_n = \frac{2}{\pi} \int_{\gamma_n} v(z)dz \geqslant 0.$$

For the defocussing cubic non-linear Schrödinger equation (a completely integrable infinite dimensional Hamiltonian system), A_n is an action variables (see [FM]). Recall the following identities from [KK1]

$$\frac{2}{\pi} \int_{\mathbb{R}} v(z)dz = \frac{1}{\pi} \iint_{\mathbb{C}} |z'(k) - 1|^2 \, dudv = \frac{1}{2}\|V\|^2, \tag{1.2}$$

where $\|V\|^2 = \int_0^1 (V_1^2(t) + V_2^2(t))dt$. Korotyaev [K1] obtained the two-sided estimates for the case ℓ^2, for example

$$\|g\|_2 \leqslant 2\|h\|_2 \leqslant \pi \|g\|_2 (2 + \|g\|_2^2), \tag{1.3}$$

$$\frac{1}{\sqrt{2}}\|g\|_2 \leqslant \|V\| \leqslant 2\|g\|_2 (1 + \|g\|_2). \tag{1.4}$$

Our main goal is to obtain similar estimates in terms of the ℓ^p and $\ell^p_\omega-$norms. Our first results are devoted to estimates in terms of ℓ^p- norms. Let below $\frac{1}{p} + \frac{1}{q} = 1, p, q \geqslant 1$.

Theorem 1.1. *Let $V \in L^1(0,1)$. Then the following estimates hold true:*

$$2^{-p}\|g\|_p \leqslant \|h\|_p \leqslant 2\|g\|_p (1 + \alpha_p^0 \ \|g\|_p^p), \qquad p \in [1,2], \quad \alpha_p^0 = \frac{2^{(p+3)p}}{\pi}, \tag{1.5}$$

$$\|h\|_p \leqslant \frac{2}{\pi} C_p^2 \|g\|_q \Big(1 + \Big(\frac{2C_p}{\pi^2}\Big)^{\frac{2}{p-1}} \|g\|_q^{\frac{2}{p-1}}\Big), \quad C_p = \Big(\frac{\pi^2}{2}\Big)^{1/p}, \quad p \geqslant 2, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.6}$$

$$\frac{\|g\|_p}{2} \leqslant \|J\|_p \leqslant \frac{2}{\sqrt{\pi}} \|g\|_p (1 + \alpha_p^0 \|g\|_p^p)^{1/2}, \qquad p \geqslant 1, \tag{1.7}$$

$$\frac{\sqrt{\pi}}{2} \|J\|_p \leqslant \|h\|_p \leqslant 4\|J\|_p (1 + \alpha_p^0 2^p \|J\|_p^p), \qquad p \geqslant 1. \tag{1.8}$$

Estimates (2.1)-(2.4) are new for $p \in [1, 2)$.

In the case $z_n^- < z_n^+$ we define the effective masses $\mu_n^\pm$ by

$$z(k) - z_n^\pm = \frac{(k - \pi n)^2}{2\mu_n^\pm} (1 + O(k - \pi n)) \qquad \text{as} \qquad z \to z_n^\pm. \tag{1.9}$$

If $|g_n| = 0$, then we set $\mu_n^\pm = 0$. Define the sequence $\mu^\pm = (\mu_n^\pm)_{n \in \mathbb{Z}}$. Our second result is

Theorem 1.2. *Let $h \in \ell^p_\omega, p \in [1, 2]$. Then the following estimates hold true*

$$\|h\|_\infty \leqslant \min\Big\{ 2\pi \|\mu^\pm\|_\infty,\ \|J\|_{p,\omega},\ 2\|g\|_{p,\omega} (1 + \alpha_p^0 \|g\|_{p,\omega}^p)^{\frac{1}{q}} \Big\}, \tag{1.10}$$

$$\|g\|_{p,\omega} \leqslant 2\|h\|_{p,\omega} \leqslant c_0^9 \|g\|_{p,\omega}, \qquad c_0 = e^{\frac{1}{\pi}\|h\|_\infty}, \tag{1.11}$$

$$\|g\|_{p,\omega} \leqslant 2\|J\|_{p,\omega} \leqslant c_0^5 2\|g\|_{p,\omega}, \tag{1.12}$$

$$\frac{\sqrt{\pi}}{2} \|J\|_{p,\omega} \leqslant \|h\|_{p,\omega} \leqslant c_0^5 \sqrt{\frac{\pi}{2}} \|J\|_{p,\omega}, \tag{1.13}$$

$$\|g\|_{p,\omega} \leqslant 2\|\mu^\pm\|_{p,\omega} \leqslant c_0^{18} \|g\|_{p,\omega}. \tag{1.14}$$

Estimates (2.5)-(2.9) are new. Introduce the real Hilbert spaces $\ell^2_{(m)}, m \in \mathbb{R}$ of the sequences $\{f_n\}_1^\infty$ equipped with the norm $\|f\|_{(m)}^2 = \sum_{n \geqslant 1} (2\pi n)^{2m} f_n^2$. Korotyaev obtained the two-sided estimates for the $\ell^2_{(m)}-$ norms, $m \geqslant 0$ for the even case $h_{-n} = h_n, n \in \mathbb{Z}$ ([K2]-[K4]) and for ℓ^2_1-norms without symmetry ([K1], ([K6])) (in all these estimates the factor $c_0 = e^{\|h\|_\infty / \pi}$ is absent).

Proposition 1.3. *Let $V \in L^2(0, 1)$. Then*

$$\|h\|_\infty \leqslant \|V\|, \tag{1.15}$$

$$\|V\|^2 \leqslant \frac{2}{\pi} \|h\|_p \, \|g\|_q, \quad p \geqslant 1, \tag{1.16}$$

$$\|V\|^2 \leqslant (\frac{2}{\pi})^{\frac{2}{p}} \|h\|_p^{\frac{2}{q}} \, \|g\|_p^{\frac{2}{p}}, \qquad p \in [1, 2], \tag{1.17}$$

$$\|V\|^2 \leqslant \frac{2}{\pi} \|h\|_\infty \, \|g\|_1 \leqslant \frac{4}{\pi^2} \|g\|_1^2, \tag{1.18}$$

$$\|h\|_\infty \leqslant \frac{2}{\pi} \|g\|_1, \qquad \|g\|_1 \leqslant 2\|h\|_1. \tag{1.19}$$

Note that the comb mappings are used in various fields of mathematics. We enumerate the more important directions:
1) the conformal mapping theory, 2) the Löwner equation and the quadratic differentials, 3) the electrostatic problems on the plane, 4) analytic capacity, 5) the spectral theory of the operators with periodic coefficients, 6) inverse problems for the Hill operator and the Dirac operator, 7) the KDV equation and the NLS equation with periodic initial value problem.

Example of an electrostatic field. Consider the system of neutral conductors $\Gamma_n, n \in \mathbb{Z}$ on the plane for some $h \in \ell^p_\omega$. In other words, we embed the system of neutral conductors Γ_n in the external homogeneous electrostatic field $E_0 = (0, -1) \in \mathbb{R}^2$ on the plane. Then on each conductor there exists the induced charge, positive $e_n > 0$ on the lower half of the conductor Γ_n and negative $(-e_n) < 0$ on the upper half of the conductor Γ_n, since their sum equals zero. As a result we have new perturbed electrostatic field $\mathcal{E} \in \mathbb{R}^2$. It is well known that $\mathcal{E} = \overline{iz'(k)} = -\nabla y(k),\ k = u + iv \in K(h),\quad z = x + iy$, where $z(k)$ is the conformal mapping from $K(h)$ onto the domain $\mathcal{Z} = \mathbb{C} \setminus \cup g_n$. The function $y(k)$ is called the potential of the electrostatic field in $K(h)$. The density of the charge on the conductor has the form $\rho(k) = |y'_u(k)|/4\pi,\ k \in \Gamma_n$ (see [LS]). Thus we obtain the induced charge e_n on the upper half of the conductor $\Gamma_n^+ = \Gamma_n \cap \mathbb{C}_+$ by:

$$e_n = \frac{1}{4\pi} \int_{\Gamma_n^+} x'_v(k) dv = \frac{1}{4\pi} |g_n|.$$

Introduce the bipolar moment d_n of the conductor Γ_n with the charge density $\rho(k)$ by $d_n = \frac{1}{4\pi} \int_{\Gamma_n} v x_v(k) dv \geqslant 0$. We transform this value into the form

$$d_n = \frac{1}{2\pi} \int_{g_n} v(x) dx = \frac{A_n}{4}.$$

In the paper [KK3] we study inverse problems for both the charge mapping $h \to e = (e_n)_{n\in\mathbb{Z}}$ and the bipolar moment mapping $h \to J$ acting in $\ell^p_\omega, p \in [1, 2]$. In order to solve the inverse problems we need a priori estimates from Theorems 1.1 and 1.2.

A priori estimates of potentials in terms of spectral data essentially simplify the proof in the inverse problems. Such simplification was introduced by Garnett and Trubowitz [GT1] and Kargaev and Korotyaev [KK2] and essentially was used in [K6]-[K9].

For the sake of the reader, we briefly recall the results existing in the literature about the a priori estimates. Firstly, we describe a priori estimates for the Hill operators $-\frac{d^2}{dt^2} + P$ in $L^2(\mathbb{R})$ with the real 1-periodic potential $P \in L^2(0, 1)$ nd $\int_0^1 P(t)dt = 0$. The spectrum of this operator consists of intervals separated by gaps $\gamma_n, n \geqslant 1$ with the lengths $|\gamma_n| \geqslant 0, n \in \mathbb{Z}$. Marchenko and Ostrovski [MO1-2] obtained the estimates: $\|P\| \leqslant C(1 + \|h\|_\infty)\|h\|_{(1)},\quad \|h\|_{(1)} \leqslant C\|P\|e^{C\|P\|}$ for some absolute constant C, where $\|P\|^2 = \int_0^1 P^2(t)dt$ and $\|h\|^2_{(1)} = \sum (2\pi n)^2 h_n^2$. These estimates are very rough since they used the Bernstein inequality. Using the harmonic measure argument Garnett and Trubowitz [GT] obtained $\|\gamma\| \leqslant (4 + \|h\|_{(1)})\|h\|_{(1)}$, where $\gamma = (|\gamma_n|)_1^\infty$ and recall that h_n are heights on the quasimomentum domain. First two-sided estimates (very rough) for g, h were obtained in [KK2].

Identities and a complete system of a priori estimates (in terms of gap lengths, effective masses etc) were obtained by Korotyaev [K1]-[K6]. In the paper [K3], [K5], [K6] the following estimates were obtained:

$$\|\gamma\|_2 \leqslant 6\|P\|(1+\|P\|^{\frac{1}{3}}), \qquad \|P\| \leqslant 4\|\gamma\|_2(1+\|\gamma\|_2^{\frac{1}{3}}),$$

$$2\|h\|_{(1)} \leqslant \pi\|P\|(1+\|P\|^{\frac{1}{3}}), \qquad \|P\| \leqslant 3(6+\|h\|_\infty)^{\frac{1}{2}}\|h\|_{(1)}.$$

These estimates show the "equivalence" of the values $\|\gamma\|, \|h\|_{(1)}, \|P\|$. Note the author solved the inverse problem and obtained the two-sided estimates for the case $P = y'$, where $y \in L^2(0,1)$. A priori two-sided estimates for the case $P^{(m)} \in L^2(0,1), m \geqslant 1$ were obtained by Korotyaev in [K1].

Secondly for the Zakharov-Shabat systems the two-sided estimates were obtained by Korotyaev for $V \in L^2(0,1)$ in [K1] (see (1.3)-(1.4)) and for $V' \in L^2(0,1)$ in [K6]. There are no a priori estimates for the case $V^{(m)} \in L^2(0,1), m \geqslant 2$. The proof of a priori estimates in [K1]-[K6] is based on the analysis of the quasimomentum as the conformal mapping.

We shortly describe the proof. In order to prove Theorem 1.1 -Proposition 1.3 we use the analysis of a conformal mapping corresponding to quasimomentum of the Zakharov-Shabat operator. That makes it possible to reformulate the problems for the differential operator as the problems of the conformal mapping theory. Then we should study the metric properties of a conformal mapping from $\mathbb{C}_+$ onto a "comb" $K_+(h)$. A similar analysis was done partially in [KK1-3], [K1-3]. In the present paper we use an approach, based on the identities for the Dirichlet integral (1.2) from [KK1] and the estimates from Theorem 2.7 and Lemma 3.1. We emphasize the important role of the Dirichlet integral (this is the energy for the conformal mapping) in this consideration.

We now describe the plan of the paper. In Section 2 we shall obtain some preliminaries results and "local basic estimates" in Theorem 2.7. In Section 3 we shall prove Lemma 3.1 and the main theorems. Moreover, we consider some examples, which describe our estimates.

2 Preliminaries

Consider a conformal mapping $z : K_+(h) \to \mathbb{C}_+$ with asymptotics $z(iv) = iv(1+o(1))$ as $v \to \infty$, where $k = u + iv \in K(h)$. Here $h = (h_n)_{n\in\mathbb{Z}} \in \ell^\infty, h_n \geqslant 0$ is some sequence and $K_+(h) = \mathbb{C}_+ \cap K(h)$ is the so-called comb domain, where $K(h)$ is given by

$$K(h) = \mathbb{C} \setminus \cup_{n\in\mathbb{Z}}\Gamma_n, \quad \Gamma_n = [u_n - ih_n, u_n + ih_n], \quad u_* = \inf_n (u_{n+1} - u_n) \geqslant 0,$$

where $u_n, n \in \mathbb{Z}$ is strongly increasing sequence of real numbers such that $u_n \to \pm\infty$ as $n \to \pm\infty$. We fix the sequence $u_n, n \in \mathbb{Z}$ and consider the conformal mapping for various $h \in \ell^\infty$. For fixed h the difference of any two such mappings equals a real constant, but the imaginary part $y(k) = \operatorname{Im} z(k)$ is unique. We call such mapping $z(k)$ the comb mapping. Define the inverse mapping $k(\cdot) : \mathbb{C}_+ \to K_+(h)$. It is clear that $k(z), z = x + iy \in \mathbb{C}_+$ has the continuous extension into $\overline{\mathbb{C}}_+$. We define "gaps" g_n, "bands" σ_n and the "spectrum" σ of the comb mapping by:

$$g_n = (z_n^-, z_n^+) = (z(u_n - 0), z(u_n + 0)), \qquad \sigma_n = [z_{n-1}^+, z_n^-], \qquad \sigma = \cup_{n\in\mathbb{Z}}\sigma_n.$$

The function $u(z) = \operatorname{Re} k(z)$ is strongly increasing on each band σ_n and $u(z) = u_n$ for all $z \in [z_n^-, z_n^+],\ n \in \mathbb{Z}$; the function $v(z) = \operatorname{Im} k(z)$ equals zero on each band σ_n and is strongly convex on each gap $g_n \neq \emptyset$ and has the maximum at some point z_n given by $v(z_n) = h_n$. If the gap is empty we set $z_n = z_n^{\pm}$. The function $z(\cdot)$ has an analytic extension (by the symmetry) from the domain $K_+(h)$ onto the domain $K(h)$ and $z(\cdot) : K(h) \to z(K(h)) = \mathcal{Z} = \mathbb{C} \setminus \cup \overline{g}_n$ is a conformal mapping. These and others properties of the comb mappings it is possible to find in the papers of Levin [Le].

We formulate our second result about the estimates for conformal mappings.

Theorem 2.1. *Let $u_* = \inf_n (u_{n+1} - u_n) > 0$. Then the following estimates hold true*

$$\|h\|_p \leqslant 2\|g\|_p(1 + \alpha_p\ \|g\|_p^p), \qquad p \in [1,2], \quad \alpha_p = \frac{(2+\pi)^p 2^{p(p+2)}}{\pi u_*^p}, \tag{2.1}$$

$$\|h\|_p \leqslant \frac{2}{\pi} C_p^2 \|g\|_q \Big(1 + \Big[\frac{2C_p}{\pi u_*}\Big]^{\frac{2}{p-1}} \|g\|_q^{\frac{2}{p-1}}\Big), \quad C_p = \Big(\frac{\pi^2}{2}\Big)^{1/p}, \quad p \geqslant 2, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{2.2}$$

$$\frac{\|g\|_p}{2} \leqslant \|J\|_p \leqslant \frac{2}{\sqrt{\pi}} \|g\|_p (1 + \alpha_p \|g\|_p^p)^{1/2}, \qquad p \geqslant 1, \tag{2.3}$$

$$\frac{\sqrt{\pi}}{2} \|J\|_p \leqslant \|h\|_p \leqslant 4\|J\|_p (1 + \alpha_p 2^p \|J\|_p^p), \qquad p \geqslant 1. \tag{2.4}$$

We formulate our second result about the estimates for conformal mappings.

Theorem 2.2. *Let $h \in \ell^p_\omega, p \in [1,2]$ and let $u_* > 0$. Then the following estimates hold true*

$$\|h\|_\infty \leqslant \min\{2\pi \|\mu^\pm\|_\infty,\ \|J\|_{p,\omega},\ 2\pi^{\frac{-1}{p}} \|g\|_{p,\omega} (1 + \alpha_p \|g\|_{p,\omega}^p)^{1/q}\}, \tag{2.5}$$

$$\|g\|_{p,\omega} \leqslant 2\|h\|_{p,\omega} \leqslant c^9 \|g\|_{p,\omega}, \qquad c = e^{\|h\|_\infty / u_*}, \tag{2.6}$$

$$\|g\|_{p,\omega} \leqslant 2\|J\|_{p,\omega} \leqslant c^5 2 \|g\|_{p,\omega}, \tag{2.7}$$

$$\frac{\sqrt{\pi}}{2} \|J\|_{p,\omega} \leqslant \|h\|_{p,\omega} \leqslant c^5 \sqrt{\frac{\pi}{2}} \|J\|_{p,\omega}, \tag{2.8}$$

$$\|g\|_{p,\omega} \leqslant 2\|\mu^\pm\|_{p,\omega} \leqslant c^{18} \|g\|_{p,\omega}. \tag{2.9}$$

Define the Dirichlet integral I_D and the moment Q_0 by

$$I_D = \frac{1}{\pi} \iint_{\mathbb{C}} |z'(k) - 1|^2\, du dv = \frac{1}{\pi} \iint_{\mathbb{C}} |k'(z) - 1|^2\, dx dy, \qquad Q_0 = \frac{1}{\pi} \int_{\mathbb{R}} v(z) dz,$$

where $k = u + iv,\ z = x + iy$. The last identity holds since the Dirichlet integral is invariant under the conformal mappings. In order to prove our main theorems we need the following

Proposition 2.3. *Let $h \in \ell^\infty$ and let $u_* \geqslant 0$. Then*

$$\frac{\|h\|_\infty^2}{2} \leqslant Q_0, \tag{2.10}$$

$$\pi Q_0 \leqslant \|h\|_p \, \|g\|_q, \quad p \geqslant 1, \tag{2.11}$$

$$I_D \leqslant (\frac{2}{\pi})^{\frac{2}{p}} \|h\|_p^{2/q} \, \|g\|_p^{2/p}, \qquad p \in [1,2], \tag{2.12}$$

$$\pi Q_0 \leqslant \|h\|_\infty \, \|g\|_1 \leqslant \frac{2}{\pi} \|g\|_1^2, \tag{2.13}$$

$$\|h\|_\infty \leqslant \frac{2}{\pi} \|g\|_1, \qquad \|g\|_1 \leqslant 2\|h\|_1. \tag{2.14}$$

Proof of Theorem 1.1- Proposition 1.3 follow directly from Theorem 2.1- Proposition 2.3 and the identity (1.2).

We recall needed results. Below we will use very often the following simple estimate

$$|g_n| \leqslant 2h_n, \quad \text{all} \quad n \in \mathbb{Z}, \tag{2.15}$$

see e.g. [MO1], [KK1]. Hence if $h \in \ell^p_\omega$, then $\gamma \in \ell^p_\omega$.

For each $h \in \ell^\infty$ the following estimates and identities hold true

$$\frac{1}{4}\|g\|^2 \leqslant 2Q_0 = I_D = \sum A_n = \|J\|^2 \leqslant \frac{2}{\pi} \sum_{n \in \mathbb{Z}} h_n |g_n|, \tag{2.16}$$

$$\max\Big\{\frac{|g_n|^2}{4}, \frac{|g_n| h_n}{\pi}\Big\} \leqslant A_n = \frac{2}{\pi} \int_{g_n} v(x) dx \leqslant \frac{2|g_n| h_n}{\pi}, \tag{2.17}$$

see [KK1]. These show that functional $Q_0 = \frac{1}{\pi} \int_{\mathbb{R}} v(x) \, dx$ is bounded for $h \in \ell^2$.

Below we will sometimes write $g_n(h), z(k,h), ..,$ instead of $g_n, z(k), ..,$ when several sequences $h \in \ell^\infty$ are being dealt with. Recall the Lindelöf principle (see [J]), which is formulated in the form, convenient for us (see [KK1]):

Let $h, \ \widetilde{h} \in \ell^\infty$; and let $\widetilde{h}_n \leqslant h_n$ for all $n \in \mathbb{Z}$. Then the following estimates hold:

$$y(k, \widetilde{h}) \geqslant y(k,h), \qquad \textit{all} \quad k \in K_+(h), \tag{2.18}$$

$$Q_0(\widetilde{h}) \leqslant Q_0(h) \qquad \textit{and if} \quad Q_0(\widetilde{h}) = Q_0(h), \quad \textit{then } \widetilde{h} = h, \tag{2.19}$$

$$|\sigma_n(\widetilde{h})| \geqslant |\sigma_n(h)|. \tag{2.20}$$

Define the effective masses ν_n in the plane $K(h)$ for the end of the slit $[u_n + ih_n, u_n - ih_n], h_n > 0$ by

$$k(z) - (u_n + ih_n) = \frac{(z - z_n)^2}{2i\nu_n}(1 + O(z - z_n))) \quad as \quad z \to z_n. \tag{2.21}$$

Thus we obtain $\nu_n = 1/|k''(z_n)|$, if $h_n > 0$ and we set $\nu_n = 0$ if $|g_n| = 0$. We show the possibility of the Lindelöf principle in the following Lemma.

Lemma 2.4. *For each $h \in \ell^\infty$ the estimate* (2.10) *and the following estimate hold true*

$$\nu_n \leqslant h_n, \qquad all \quad n \in \mathbb{Z}. \tag{2.22}$$

Proof. Sufficiently to proof for the case $n = 0$. We apply estimate (2.18) to h and to the new sequence: $\widetilde{h}_0 = h_0$ and $\widetilde{h}_n = 0$ if $n \neq 0$. It is clear that $z(k, \widetilde{h}) = \sqrt{(k - u_0)^2 + h_0^2}$ (the principal value). Then (2.18) gives

$$y(k, h) \leqslant \operatorname{Im}(\sqrt{(k - u_n)^2 + h_0^2}), \ k \in K_+(h).$$

Then asymptotics (2.21) of the function $z(k, h)$ as $k \to u_0 + ih_0$ yields (2.22). In order to prove (2.10) we use (2.19) since $Q_0(\widetilde{h}) = h_0^2/2$. Note that it and (2.19) yield (2.10). ■

We recall estimates from [K4].

Theorem 2.5. *Let $h \in \ell^\infty$. Then for any $r > 0, n \in \mathbb{Z}$ the following estimate holds true*

$$h_n^2 \leqslant \frac{\pi}{4} \max\Big\{1, \frac{h_n}{r}\Big\} \iint_{u_n + S_r} |z'(k) - 1|^2 du dv, \quad S_r = \{z \in \mathbb{C} : \ |\operatorname{Re} z| < r\}, \tag{2.23}$$

If in addition, $\inf_n (u_{n+1} - u_n) = u_* > 0$, *then*

$$\frac{\pi}{4} I_D \leqslant \|h\|_2^2 \leqslant \frac{\pi^2}{2} \max\Big\{1, \frac{\|h\|_\infty}{u_*}\Big\} I_D \leqslant \frac{\pi^2}{2} \max\Big\{1, \frac{I_D^{1/2}}{u_*}\Big\} I_D, \tag{2.24}$$

$$\frac{1}{2}\|g\|_2 \leqslant \|h\|_2 \leqslant \pi \|g\|_2 \Big(1 + \frac{2}{u_*^2}\|g\|_2^2\Big), \tag{2.25}$$

$$\frac{\|g\|_2}{2} \leqslant \|J\|_2 \leqslant \sqrt{2}\|g\|_2 (1 + \frac{\sqrt{2}}{u_*}\|g\|). \tag{2.26}$$

In order to prove Theorem 2.7 we need the following result about the simple mapping and the domain $S_r = \{z \in \mathbb{C} : \ |\operatorname{Re} z| < r\}, \ r > 0$.

Lemma 2.6. *The function $f(k) = \sqrt{k^2 + h^2}, \ k \in \mathbb{C} \setminus [-ih, ih], h > 0$ is the conformal mapping from $\mathbb{C} \setminus [-ih, ih]$ onto $\mathbb{C} \setminus [-h, h]$ and $S_r \setminus [-h, h] \subset f(S_r \setminus [-ih, ih])$ for any $r > 0$.*

Proof. Consider the image of the half-line $k = r + iv, v > 0$. We have the equations

$$x^2 + y^2 = \xi \equiv r^2 + h^2 - v^2, \qquad xy = rv. \tag{2.27}$$

The second identity in (2.27) yields $x > 0$ since $y > 0$. Then $x^4 - \xi x^2 - r^2 v^2 = 0$, and enough to check the following inequality $x^2 = \frac{1}{2}(\xi + \sqrt{\xi^2 + 4r^2 v^2}) > r^2$. The last estimate follows from the simple relations

$$(r^2 + h^2 - v^2)^2 + 4r^2 v^2 > (r^2 + v^2 - h^2)^2, \qquad 4r^2 v^2 > 4r^2(v^2 - h^2). \quad ■$$

We prove the local estimates for the small slits, which are crucial for us.

Theorem 2.7. *Let $h \in \ell^\infty$. Assume that $(u_n - r, u_n + r) \subset (u_{n-1}, u_{n+1})$ and $h_n \leqslant \frac{r}{2}$, for some $n \in \mathbb{Z}$ and $r > 0$. Then*

$$|h_n - |\mu_n^\pm|| \leqslant \frac{2+\pi}{r}|\mu_n^\pm|\sqrt{I_n}, \quad I_n = \frac{1}{\pi}\iint_{u_n+S_r}|z'(k)-1|^2 dudv, \tag{2.28}$$

$$0 \leqslant h_n - \nu_n \leqslant 2\frac{2+\pi}{r}h_n\sqrt{I_n}, \tag{2.29}$$

$$0 \leqslant h_n - \frac{|g_n|}{2} \leqslant \frac{2+\pi}{r}h_n\sqrt{I_n}. \tag{2.30}$$

Proof. Define the functions $f(k) = \sqrt{k^2 + h_n^2}$, $k \in S_r \setminus [-ih_n, ih_n], \phi = f^{-1}$ and $F(w) = z(u_n + \phi(w), h), w = p + iq$, where the variable $w \in G_1 = f((S_r \setminus [-ih_n, ih_n]))$. The function F is real for real w, then F is analytic in the domain $G = G_1 \cup [-h_n, h_n]$ and Lemma 2.6 yields $S_r \subset G$. Let now $|w_1| = \frac{r}{2}$ and $B_r = \{z : |z| < r\}$. Then the following estimates hold

$$\sqrt{\pi}\frac{r}{2}|F'(w_1) - 1| \leqslant (\iint_{B_r}|F'(w)-1|^2 dpdq)^{1/2} \leqslant \tag{2.31}$$

$$\leqslant \Big(\iint_{B_r}|(F(w)-\phi(w))'|^2 dpdq\Big)^{1/2} + \Big(\iint_{B_r}|\phi'(w)-1|^2 dpdq\Big)^{1/2}.$$

The invariance of the Dirichlet integral with respect to the conformal mapping gives

$$\iint_{B_r}|(F(w)-\phi(w))'|^2 dpdq = \iint_{\phi(B_r)}|z'(k)-1|^2 dudv \leqslant \iint_{S_r+u_n}|z'(k)-1|^2 dudv = \pi I_n. \tag{2.32}$$

Moreover, the identity $2Q_0 = I_D$ implies

$$\frac{1}{\pi}\iint_{B_r}|\phi'(w)-1|^2 dpdq \leqslant \frac{1}{\pi}\iint_{\mathbb{C}}|\phi'(w)-1|^2 dpdq = \frac{2}{\pi}\int_{-h_n}^{h_n}\sqrt{h_n^2 - x^2}\,dx = h_n^2. \tag{2.33}$$

Then (2.31)-(2.33) for $|w_1| = \frac{r}{2}$ yields $|F'(w_1) - 1| \leqslant \frac{2}{r}(\sqrt{I_n} + h_n)$, and (2.23) gives

$$h_n^2 \leqslant \frac{\pi^2}{4}\cdot\frac{1}{\pi}\iint_{S_r+u_n}|z'(k)-1|^2 dudv \leqslant \frac{\pi^2}{4}I_n.$$

Then for $|w_1| = \frac{r}{2}$ we have

$$|F'(w_1) - 1| \leqslant \frac{2}{r}(1+\frac{\pi}{2})\sqrt{I_n} = \frac{2+\pi}{r}\sqrt{I_n}, \tag{2.34}$$

and the maximum principle yields the needed estimates for $|w_1| \leqslant r/2$.

We prove (2.28) for μ_n^+. The definition of $\mu_n^\pm$ (see (1.9)) implies

$$F'(h_n) = \lim_{x \searrow h_n} z'(u_n + g(x)) \cdot g'(x) = \lim_{x \searrow h_n}\frac{g(x)}{\mu_n^+}\cdot\frac{x}{g(x)} = \frac{h_n}{\mu_n^+}.$$

The substitution of the last identity into (2.34) gives (2.28). The proof for μ_n^- is similar.

We show (2.29). The definition of ν_n (see (2.21)) yields

$$(z(k)-z_n)^2 = 2i\nu_n(k-u_n-ih_n)(1+o(1)) \ \ as \ \ k\to u_n+ih_n,$$

$$\phi(w)-ih_n = -\frac{i}{2h_n}(w-z_n)^2(1+o(1)) \ \ as \ w\to u_n.$$

Then we have $F'(0)=\sqrt{\frac{\nu_n}{h_n}}$ and the substitution of the last identity into (2.34) shows $\left|\sqrt{\frac{\nu_n}{h_n}}-1\right| \leqslant \frac{2+\pi}{r}\sqrt{I_n}$, which gives (2.29), since by (2.17), $\nu_n\leqslant h_n$. Estimate (2.34) yields

$$0\leqslant 2h_n-|g_n| = \int_{-h_n}^{h_n}(1-F'(x))dx \leqslant \frac{2h_n}{r}(2+\pi)\sqrt{I_n},$$

which implies (2.30). ■

We prove the estimates in terms of the ℓ^p-norms.

Proof of Proposition 2.3. We have proved (2.10) in Lemma 2.4. Estimate (2.16) and the Hölder inequality yield (2.11). Using (2.10), (2.16) and the Hölder inequality, we obtain

$$(\pi/2)I_D=\pi Q_0\leqslant \sum h_n|g_n|\leqslant \|h\|_\infty^{1-\frac{p}{q}}\sum_{n\in\mathbb{Z}}|g_n|\,h_n^{p/q}\leqslant (I_D)^{(1-\frac{p}{q})/2}\|g\|_p\|h\|_p^{\frac{p}{q}},$$

$$(\pi/2)I_D^{\frac{(1+\frac{p}{q})}{2}}\leqslant \|g\|_p\|h\|_p^{\frac{p}{q}}, \qquad \text{and} \qquad I_D\leqslant\Big(\frac{2}{\pi}\Big)^{\frac{2}{p}}\|g\|_p^{\frac{2}{p}}\|h\|_p^{\frac{2}{q}}.$$

Estimate (2.11) at $q=1$ implies the first one in (2.13). The last result and (2.10) yield the first inequality in (2.14) and then the second one in (2.13). The second estimate in (2.14) follows from $|g_n|\leqslant 2h_n,\ n\in\mathbb{Z}$ (see (2.15)). ■

3 Proof of the mains theorems

Proof of Theorem 2.1. Let $p\in[1,2]$ and $r=\frac{u_*}{2}$. Estimate (2.23) implies

$$h_n^2\leqslant\frac{\pi^2}{4}\max\{1,\frac{h_n}{r}\}I_n, \quad I_n=\frac{1}{\pi}\iint_{u_n+S_r}|z'(k)-1|^2dudv, \tag{3.1}$$

Hence

$$h_n\leqslant\frac{\pi^2}{u_*}I_n, \quad if \quad h_n>\frac{u_*}{4}, \ \ and \ \ h_n\leqslant\frac{\pi}{2}\sqrt{I_n}, \quad if \quad h_n\leqslant\frac{u_*}{4}. \tag{3.2}$$

Moreover, (2.30) yields

$$h_n\leqslant\frac{|g_n|}{2}+2\frac{2+\pi}{u_*}h_n\sqrt{I_n}, \qquad if \quad h_n<\frac{u_*}{4}, \tag{3.3}$$

and then

$$h_n\leqslant 2\pi\frac{2+\pi}{u_*}I_n, \qquad if \quad h_n<\frac{u_*}{4},\ |g_n|\leqslant h_n, \tag{3.4}$$

since $h_n \leqslant \frac{\pi}{2}\sqrt{I_n}$. Hence using (3.4), (3.2), we obtain

$$if \quad |g_n| \leqslant h_n \quad \Rightarrow \quad h_n \leqslant 2\pi \frac{2+\pi}{u_*} I_n = C_1 I_n, \qquad C_1 = 2\pi \frac{2+\pi}{u_*}.$$

The last inequality and (2.10) yield

$$\|h\|_p \leqslant (\sum_{h_n<|g_n|} h_n^p)^{\frac{1}{p}} + (\sum_{|g_n|\leqslant h_n} h_n h_+^{p-1})^{\frac{1}{p}} \leqslant \|g\|_p + C_1^{\frac{1}{p}} I_D^{\frac{p+1}{2p}}, \quad h_+ = \|h\|_\infty. \tag{3.5}$$

If we assume that $C_1^{\frac{1}{p}} I_D^{\frac{p+1}{2p}} \leqslant \|g\|_p$, then we obtain $\|h\|_p \leqslant 2\|g\|_p$.

Conversely, if we assume that $\|g\|_p \leqslant C_1^{\frac{1}{p}} I_D^{\frac{p+1}{2p}}$, then (3.5), (2.12) implies

$$\|h\|_p \leqslant 2C_1^{\frac{1}{p}} \Big[\Big(\frac{2}{\pi}\Big)^{\frac{2}{p}} \|h\|_p^{\frac{2}{q}} \|g\|_p^{2/p} \Big]^{\frac{p+1}{2p}}.$$

Hence

$$\|h\|_p^{1/p^2} \leqslant 2C_1^{\frac{1}{p}} \Big[(2/\pi)\|g\|_p \Big]^{\frac{p+1}{p^2}} \qquad \Rightarrow \qquad \|h\|_p \leqslant 2^{p^2} C_1^p (2/\pi)^{p+1} \|g\|_p^{1+p},$$

which yields (2.1).

Let $p \geqslant 2$. Using inequality (2.24), (2.10) we obtain

$$\|h\|_p \leqslant (\sum h_+^{p-2} h_n^2)^{\frac{1}{p}} \leqslant C_p b^{\frac{1}{p}} I_D^{\frac{1}{2}}, \qquad b = b(h_+),\ b(t) = \max\{1, \frac{t}{r}\},\ h_+ = \|h\|_\infty. \tag{3.6}$$

Consider the case $b \leqslant 1$. Then (3.6), (2.11) imply

$$\|h\|_p^2 \leqslant C_p^2 I_D \leqslant C_p^2 (2/\pi) \|h\|_p \|g\|_q, \qquad \text{and} \qquad \|h\|_p \leqslant (2C_p^2/\pi)\|g\|_q,$$

Consider the case $b > 1$. Then the substitution of (2.10), (2.11) into (3.6) yield

$$\|h\|_p \leqslant C_p I_D^{\frac{p+1}{2p}} u_*^{-1/p} \leqslant u_*^{-1/p} C_p [(2/\pi)\|h\|_p \|g\|_q]^{\frac{p+1}{2p}},$$

and

$$\|h\|_p^{\frac{p-1}{2p}} \leqslant C_p u_*^{-1/p} [(2/\pi)\|g\|_q]^{\frac{p+1}{2p}} \qquad \Rightarrow \qquad \|h\|_p \leqslant (C_p u_*^{-1/p})^{\frac{2p}{p-1}} (2/\pi)^{\frac{p+1}{p-1}} \|g\|_q^{\frac{p+1}{p-1}},$$

and combining these two cases we have (2.2).

Estimate $|g_n| \leqslant 2J_n$ (see (2.17)) yields the first one in (2.3). Relation (2.17) implies

$$\|J\|_p^p = \sum |J_n|^p \leqslant \sum (2/\pi)^{p/2} h_n^{p/2} |g_n|^{p/2} \leqslant (2/\pi)^{p/2} \|h\|_p^{p/2} \|g\|_p^{p/2},$$

and using (2.1) we obtain the second estimate in (2.3):

$$\|J\|_p \leqslant \sqrt{2/\pi} \|g\|_p^{1/2} [2\|g\|_p (1 + \alpha_p \|g\|_p^p)]^{1/2} = \frac{2}{\sqrt{\pi}} \|g\|_p (1 + \alpha_p \|g\|_p^p)^{1/2},$$

recall that $\alpha_p = (2^{p+2}(2+\pi)/u_*)^p/\pi$. Inequality $J_n^2 \leqslant 4h_n^2/\pi$ (see (2.17)) yields the first estimate in (2.4). Using (2.1) and $\|g\|_p \leqslant 2\|J\|_p$ (see (2.3)) we deduce that

$$\|h\|_p \leqslant 2\|g\|_p(1+\alpha_p\|g\|_p^p) \leqslant 4\|J\|_p(1+\alpha_p 2^p\|J\|_p^p). \quad \blacksquare$$

Recall the following identity for $v(z) = \operatorname{Im} k(z), z = x+iy$ from [KK1]:

$$v(x) = v_n(x)(1+Y_n(x)), \quad Y_n(x) = \frac{1}{\pi}\int_{\mathbb{R}\setminus g_n} \frac{v(t)dt}{|t-x|\, v_n(t)}, \quad v_n(x) = |(x-z_n^+)(x-z_n^-)|^{\frac{1}{2}}, \quad (3.7)$$

for all $x \in g_n = (z_n^-, z_n^+)$. In order to prove Theorem 2.2 we need the following results.

Lemma 3.1. *Let $h \in \ell^\infty$ and $u_* > 0$ and $c = e^{\frac{\|h\|_\infty}{u_*}}$. Then the following estimates hold:*

$$s = \inf|\sigma_n| \leqslant u_* \leqslant \frac{\pi s}{2}\max\Big\{e^2, c^{\frac{5\pi}{2}}\Big\}, \quad (3.8)$$

$$1 + \frac{2\|h\|_\infty}{s\pi} \leqslant c^9, \quad (3.9)$$

$$\max_{n\in g_n} Y_n(x) \leqslant \frac{2\|h\|_\infty}{\pi s}, \quad n \in \mathbb{Z}, \quad (3.10)$$

$$2h_n \leqslant |g_n|(1+\max_{n\in g_n} Y_n(x)) \leqslant |g_n|(1+\frac{2\|h\|_\infty}{\pi s}) \leqslant |g_n|c^9, \quad n \in \mathbb{Z}. \quad (3.11)$$

Proof. Introduce the domain

$$G = \{z \in \mathbb{C} : h_+ \geqslant \text{Im } z > 0, \ \text{Re } z \in (-\frac{u_*}{2}, \frac{u_*}{2})\} \cup \{\text{Im } z > h_+\}, \quad h_+ = \|h\|_\infty.$$

Let F be the conformal mapping from G onto $\mathbb{C}_+$, such that $F(iy) \sim iy$ as $y \nearrow +\infty$ and let α, β be images of the points $\frac{u_*}{2}, \frac{u_*}{2} + ih_+$ respectively. Define the function $f = \operatorname{Im} F$. Fix any $n \in \mathbb{Z}$. Then the maximum principle yields

$$y(k) = \operatorname{Im} z(k,h) \geqslant f(k-p_n), \quad k \in G + p_n, \quad p_n = \frac{1}{2}(u_{n-1}+u_n).$$

Due to the fact that these positive functions equal zero on the interval $(p_n - \frac{u_*}{2}, p_n + \frac{u_*}{2})$, we obtain

$$\partial_v y(x) = \partial_u x(x) \geqslant \partial_v f(x-p_n), \qquad x \in (p_n - \frac{u_*}{2}, p_n + \frac{u_*}{2}).$$

where $\partial_x = \frac{\partial}{\partial x}$. Then

$$z(u_n) - z(u_{n-1}) \geqslant \int_{-u_*/2}^{u_*/2} \partial_v f(x)\, dx = 2\alpha > 0,$$

and the estimate $s \leqslant u_*$ (see [KK1]) implies $2\alpha \leqslant s \leqslant u_*$. Let $w : \mathbb{C}_+ \to G$ be the inverse function for F, which is defined uniquely and the Christoffel-Schwartz formula yields

$$w(z) = \int_0^z \sqrt{\frac{t^2-\beta^2}{t^2-\alpha^2}}dt, \qquad 0<\alpha<\beta.$$

Then we have

$$\frac{u_*}{2} = \int_0^\alpha \sqrt{\frac{\beta^2-t^2}{\alpha^2-t^2}}dt, \quad h_+ = \int_\alpha^\beta \sqrt{\frac{\beta^2-t^2}{t^2-\alpha^2}}dt. \tag{3.12}$$

The first integral in (3.12) has the simple double-sided estimates

$$\alpha = \int_0^\alpha dt \leqslant \frac{u_*}{2} \leqslant \int_0^\alpha \frac{\beta dt}{\sqrt{\alpha^2-t^2}} = \frac{\beta\pi}{2},$$

that is

$$2\alpha \leqslant s \leqslant u_* \leqslant \pi\beta. \tag{3.13}$$

Consider the second integral in (3.12). Let $\varepsilon = \beta/\alpha \geqslant 5$ and using the new variable $t = \alpha\cosh r, \cosh\delta = \varepsilon$, we obtain

$$h_+ = \alpha\int_0^\delta \sqrt{\varepsilon^2-\cosh^2 r}dr \geqslant \alpha\varepsilon\int_0^{\delta/2}\sqrt{1-\frac{\cosh^2 r}{\varepsilon^2}}dr \geqslant \beta\delta\frac{2}{5},$$

since for $r \leqslant \delta/2$ we have the simple inequality

$$\frac{\cosh^2 r}{\cosh^2\delta} \leqslant e^{-\delta}(1+e^{-\delta})^2 \leqslant \varepsilon^{-1}(1+\varepsilon^{-1})^2.$$

Due to $\varepsilon \leqslant e^\delta$ we get $\varepsilon \leqslant \exp(5h_+/2\beta)$ and estimate (3.13) implies

$$\frac{1}{s} \leqslant \frac{\pi}{2u_*}\exp(\frac{5\pi}{2u_*}h_+), \quad if \quad \varepsilon \geqslant 5. \tag{3.14}$$

If $\varepsilon \leqslant 5$, then using (3.13) again we obtain

$$\frac{1}{s} \leqslant \frac{\varepsilon}{2\beta} \leqslant \frac{\pi\varepsilon}{2u_*} \leqslant \frac{\pi}{2u_*}5, \qquad if \quad \varepsilon \leqslant 5.$$

and the last estimate together with (3.14) yield (3.8),(3.9).

Identity (3.7) for $x \in g_n = (z_n^-, z_n^+)$ implies

$$\pi Y_n(x) = \int_{-\infty}^{z_n^- - s}\frac{v(t)dt}{|t-x|v_n(t)} + \int_{z_n^+ + s}^\infty \frac{v(t)dt}{|t-x|v_n(t)} \leqslant \int_{-\infty}^{z_n^- - s}\frac{h_+dt}{|t-z_n^-|^2} + \int_{z_n^+ + s}^\infty \frac{h_+dt}{|t-z_n^+|^2} \leqslant \frac{2h_+}{s}.$$

Using (3.7), (3.9) and simple inequality $v_n(z_n) \leqslant |g_n|/2$ we have (3.11). ■

We prove the two-sided estimates of $h_n, |g_n|, \mu_n^\pm, J_n$ in the weight spaces.

Proof of Theorem 2.2. The first estimate in (2.5) follows from $h_n \leqslant 2\pi|\mu_n^\pm|$ (see [KK1]). The second one in (2.5) follows from $\|h\|_\infty \leqslant \sqrt{I_D} = \|J\|_2 \leqslant \|J\|_p \leqslant \|J\|_{p,\omega}$ since $\omega_n \geqslant 1$ for any $n \in \mathbb{Z}$. Moreover, substituting (2.12) into $\|h\|_\infty \leqslant \sqrt{I_D}$, using (2.1) and $\|f\|_p \leqslant \|f\|_{p,\omega}$ for any f, we obtain the last estimate in (2.5).

Recall that $c = \exp \frac{\|h\|_\infty}{u_*}$. The first estimate in (2.6) follows from (2.15). Due to (3.11) we get $2h_n \leqslant c^9|g_n|$, which yields the second estimate in (2.6).

The first estimate in (2.7) follows from (2.17). Using (2.17), (3.11) we have $J_n^2 \leqslant 2|g_n|h_n/\pi \leqslant (c^9/\pi)|g_n|^2$, which gives the second estimate in (2.7).

The first estimate in (2.8) follows from (2.17), (2.15). Using (2.17), (3.11) we obtain $h_n^2 \leqslant c^9|g_n|h_n/2 \leqslant (\pi c^9/2)J_n^2$, which yields the second inequality in (2.8).

Identity $2|\mu_n^\pm| = |g_n|[1 + Y_n(z_n^\pm)]^2$ (see [KK1]) implies $2|\mu_n^\pm| \geqslant |g_n|$, which yields the first inequality in (2.9). Moreover, using (3.11) we obtain the estimate $2|\mu_n^\pm| \leqslant c^{18}|g_n|$, which gives the second one in (2.9). ■

Recall that for a compact subset $\Omega \subset \mathbb{C}$ the analytic capacity is given by

$$\mathcal{C} = \mathcal{C}(\Omega) = \sup\Big[|f'(\infty)| : f \textit{ is analytic in } \mathbb{C} \setminus \Omega; \quad |f(k)| \leqslant 1, \ k \in \mathbb{C} \setminus \Omega\Big], \tag{3.15}$$

where $f'(\infty) = \lim\limits_{|k|\to\infty} k(f(k) - f(\infty))$. We will use the well known Theorem (see [Iv], [Po])

Theorem *(Ivanov-Pommerenke). Let $E \subset \mathbb{R}$ be compact. Then the analytic capacity $\mathcal{C}(E) = |E|/4$, where $|E|$ is the Lebesgue measure (the length) of the set E. Moreover, the Ahlfors function f_E (the unique function, which gives* sup *in the definition of the analytic capacity) has the following form:*

$$f_E(z) = \frac{\exp\left(\frac12 \phi_E(z)\right) - 1}{\exp\left(\frac12 \phi_E(z)\right) + 1}, \qquad \phi_E(z) = \int_E \frac{dt}{z-t}; \quad z \in \mathbb{C} \setminus E. \tag{3.16}$$

We will use the following simple remark: Let $S_1, S_2, \ldots, S_N$ be disjoint continua in the plane $\mathbb{C}$; $D = \mathbb{C}\setminus\cup_{n=1}^N S_n$. Introduce the class $\Sigma'(D)$ of the conformal mapping w from the domain D onto $\mathbb{C}$ with the following asymptotics: $w(k) = k + [Q(w) + o(1)]/k, \ k \to \infty$. If $\Omega \subset \mathbb{C}$ is compact; $D = \mathbb{C} \setminus \Omega, \ g \in \Sigma'(D)$, then $\mathcal{C}(\Omega) = \mathcal{C}(\mathbb{C} \setminus g(D))$. It follows immediately from the definition of the analytic capacity.

Let $\ell^2_{fin} \subset \ell^2$ be the subset of finite sequences of non negative numbers. Then, using the Ivanov-Pommerenke Theorem and the last remark we obtain

$$\|g(h)\|_1 = \mathcal{C}(\Gamma(h)), \quad \textit{where} \quad h \in \ell^2_{fin}, \ \Gamma(h) = \cup[u_n - ih_n, u_n + ih_n];$$

Now we estimate the Dirichlet integral $I_D(h) = 2Q_0(h)$ for the case $u_* \geqslant 0$, using a geometric construction.

Theorem 3.2. *Let $h \in \ell^\infty, h_n \to 0$ as $|n| \to \infty$; and let $\widetilde{h} = \widetilde{h}(h)$ be given by*

if $h = 0$, then $\widetilde{h} = 0$,

if $h \neq 0$, let an integer n_1 be such that $\widetilde{h}_{n_1} = h_{n_1} = \max_{n\in\mathbb{Z}} h_n > 0$; assume that the numbers $h_{n_1}, h_{n_2}, \ldots, h_{n_k}$ are defined, then n_{k+1} is given by

$$\widetilde{h}_{n_{k+1}} = h_{n_{k+1}} = \max_{n\in B} h_n > 0, \quad B = \{n \in \mathbb{Z} : |u_n - u_{n_s}| > h_{n_s}, 1 \leqslant s \leqslant k\}, \tag{3.17}$$

and let $\widetilde{h}_n = 0$, *if* $n \notin \{n_k, k \in \mathbb{Z}\}$.

Then the following estimates hold:

$$\frac{1}{\pi^2}\|\widetilde{h}\|_2^2 \leqslant Q_0(h) = \frac{I_D(h)}{2} \leqslant \frac{2\sqrt{2}}{\pi}\|\widetilde{h}\|_2^2. \tag{3.18}$$

Proof. The Lindelöf principal yields $Q_0(\widetilde{h}) \leqslant Q_0(h)$. On the other hand open squares $P_k = (u_{n_k} - t_k, u_{n_k} + t_k) \times (-t_k, t_k), t_k \equiv h_{n_k} = \widetilde{h}_{n_k}, k \in Z$, does not overlap. Then applying (2.23) to the function $(z(k, \widetilde{h}) - k)$ and P_k, we obtain

$$2t_k^2 \leqslant \pi \int\int_{P_k} |z'(k, \widetilde{h}) - 1|^2 \, dudv, \tag{3.19}$$

and

$$Q_0(\widetilde{h}) = \frac{1}{2} I_D(\widetilde{h}) \geqslant \frac{1}{\pi^2} \sum_{k \geqslant 1} t_k^2 = \frac{1}{\pi^2}\|\widetilde{h}\|_2^2$$

which yields the first estimate in (3.18).

Let $\Omega_k = \{n \in \mathbb{Z} : u_n \in [u_{n_k} - t_k, u_{n_k} + t_k]\}$. By the Lindelöf principal, the gap length $|g_{n_k}|$ such that $[u_n, u_n + ih_n], n \in \Omega_k$ increases if we take off all another slits. By the Ivanov-Pomerenke Theorem (see above), the sum of new gap lengths equals to $4 \times$ *capacity* of the set $E = \cup_{n \in \Omega_k}[u_n - h_n, u_n + h_n]$, which is less than the diameter of the set E. Then $\sum_{n \in \Omega_k} |g_n(h)| \leqslant 2\sqrt{2} t_k$, and using the last estimate we obtain

$$\pi Q_0(h) \leqslant \sum_{n \in \mathbb{Z}} h_n |g_n| \leqslant \sum_{k \geqslant 1} \sum_{n \in \Omega_k} h_n |g_n| \leqslant \sum_{k \geqslant 1} t_k \sum_{n \in \Omega_k} |g_n| \leqslant 2\sqrt{2} \sum_{k \geqslant 1} t_k^2 = 2\sqrt{2}\|\widetilde{h}\|_2^2, \tag{3.20}$$

since $h_n \leqslant t_k, n \in \Omega_k$ and the diameter of the set E is less than or equals $2\sqrt{2} t_k$. ■

Note that the proved Theorem shows that estimates (2.25),(2.26) hold true for the weaker conditions on the sequence $u_n, n \in \mathbb{Z}$.

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